Statistical Methods for Analysis with Missing Data

Lecture 7 setup: example of EM algorithm

Mauricio Sadinle

Department of Biostatistics

UNIVERSITY *of* WASHINGTON

Previous Lecture

The EM algorithm for ML estimation with missing data:

- General derivation of the algorithm
- The EM algorithm under MAR
  - The Expectation step:

$$
\begin{aligned}
Q_\theta(\theta \mid \theta^{(t)}) &= E\ [\ \log p(z_{(r)}, Z_{(\bar{r})} \mid \theta)\ \mid\ Z_{(r)} = z_{(r)}, \theta^{(t)}] \\
&= \int_{\mathcal{Z}_{(\bar{r})}} p(z_{(\bar{r})} \mid z_{(r)}, \theta^{(t)}) \log p(z \mid \theta)\ dz_{(\bar{r})}
\end{aligned}
$$

  - The Maximization step:

$$
\theta^{(t+1)} = \arg\max_\theta\ Q_\theta(\theta \mid \theta^{(t)})
$$

- Example with two categorical variables

Today's Lecture

- Continued example of EM for two categorical variables
- Coding it in R
- Assessing variability of estimates via the bootstrap

Example of EM Algorithm Under MAR

- Let $Z_i = (Z_{i1}, Z_{i2})$, $\quad Z_{i1}, Z_{i2} \in \{1,2\}$, $\quad Z_i$'s are i.i.d.
- Therefore

$$p(Z_{i1} = z_{i1}, Z_{i2} = z_{i2} \mid \theta) = \pi_{z_{i1} z_{i2}},$$

- The likelihood of the study variables is

$$\begin{aligned} L(\theta) &= \prod_i \pi_{z_{i1} z_{i2}} \\ &= \prod_i \left[\prod_{k,l} \pi_{kl}^{I(z_{i1}=k, z_{i2}=l)} \right] \\ &= \prod_i \left[\prod_{k,l} \pi_{kl}^{W_{ikl}} \right] \end{aligned}$$

where

$$W_{ikl} = I(Z_{i1} = k, Z_{i2} = l), \quad k, l \in \{1,2\}$$

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Example of EM Algorithm Under MAR

- Let $R_i = (R_{i1}, R_{i2})$, $R_{i1}, R_{i2} \in \{0,1\}$, R_i's are i.i.d.
- Denote the ith realized value as $r_i = (r_{i1}, r_{i2})$
- Note that if $r_i = 10$ and $z_{i1} = k$ we observe

$$W_{ik+} \equiv \sum_l W_{ikl} = \sum_l I(Z_{i1} = k, Z_{i2} = l) = I(Z_{i1} = k)$$

- If $r_i = 01$ and $z_{i2} = l$ we observe

$$W_{i+l} \equiv \sum_k W_{ikl} = \sum_k I(Z_{i1} = k, Z_{i2} = l) = I(Z_{i2} = l)$$

- We'll similarly denote

$$\pi_{k+} = \sum_l \pi_{kl}, \quad \text{and} \quad \pi_{+l} = \sum_k \pi_{kl}$$

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Q_\theta(\theta \mid \theta^{(t)}) &= \sum_i E\left[\ \log p(z_{i(r_i)}, Z_{i(\bar{r}_i)} \mid \theta)\ \mid\ Z_{i(r_i)} = z_{i(r_i)}, \theta^{(t)}\right] \\
&= \sum_i \sum_r I(r_i = r) E\left[\ \log p(z_{i(r)}, Z_{i(\bar{r})} \mid \theta)\ \mid\ Z_{i(r)} = z_{i(r)}, \theta^{(t)}\right] \\
&= \sum_i \Bigg[\ I(r_i = 11) \log \pi_{z_{i1} z_{i2}} + \\
&\qquad I(r_i = 10) \sum_l \frac{\pi^{(t)}_{z_{i1} l}}{\pi^{(t)}_{z_{i1}+}} \log \pi_{z_{i1} l} + \\
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Note that in the above expression we can rewrite as follows:

- For $r = 11$

$$\log \pi_{z_{i1} z_{i2}} = \sum_{k,l} I(z_{i1} = k, z_{i2} = l) \log \pi_{kl}$$

- For $r = 10$

$$\sum_{l} \frac{\pi^{(t)}_{z_{i1} l}}{\pi^{(t)}_{z_{i1}+}} \log \pi_{z_{i1} l} = \sum_{k,l} I(z_{i1} = k) \frac{\pi^{(t)}_{kl}}{\pi^{(t)}_{k+}} \log \pi_{kl}$$

- For $r = 01$

$$\sum_{k} \frac{\pi^{(t)}_{k z_{i2}}}{\pi^{(t)}_{+z_{i2}}} \log \pi_{k z_{i2}} = \sum_{k,l} I(z_{i2} = l) \frac{\pi^{(t)}_{kl}}{\pi^{(t)}_{+l}} \log \pi_{kl}$$

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$$\sum_{k,l} \pi^{(t)}_{kl} \log \pi_{kl}$$

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- For $r = 00$

$$\sum_{k,l} \pi^{(t)}_{kl} \log \pi_{kl}$$

Example of EM Algorithm Under MAR

Let's define the "predicted" W_{ikl} at the $(t+1)$th iteration as

$$W_{ikl}^{(t+1)} = W_{ikl} I(r_i = 11) \; + \; W_{ik+} \frac{\pi_{kl}^{(t)}}{\pi_{k+}^{(t)}} I(r_i = 10) \; + \; W_{i+l} \frac{\pi_{kl}^{(t)}}{\pi_{+l}^{(t)}} I(r_i = 01) \; + \; \pi_{kl}^{(t)} I(r_i = 00)$$

Example of EM Algorithm Under MAR

With this notation

- E step:

$$Q_\theta(\theta \mid \theta^{(t)}) = \sum_{i,k,l} W_{ikl}^{(t+1)} \log \pi_{kl}$$

- M step:

$$\pi_{kl}^{(t+1)} = \frac{1}{n} \sum_i W_{ikl}^{(t+1)}$$

Example of EM Algorithm Under MAR

Or equivalently,

- E step:

$$W_{ikl}^{(t+1)} = W_{ikl} I(r_i = 11) \; + \; W_{ik+} \frac{\pi_{kl}^{(t)}}{\pi_{k+}^{(t)}} I(r_i = 10) \; + \; W_{i+l} \frac{\pi_{kl}^{(t)}}{\pi_{+l}^{(t)}} I(r_i = 01) \; + \; \pi_{kl}^{(t)} I(r_i = 00)$$

- M step:

$$\pi_{kl}^{(t+1)} = \frac{1}{n} \sum_i W_{ikl}^{(t+1)}$$

Example of EM Algorithm Under MAR

Note that the M step can be written as:

$$\pi_{kl}^{(t+1)} = \frac{1}{n} \sum_i W_{ikl}^{(t+1)}$$
$$= \frac{1}{n} \left(n_{11kl} + \frac{\pi_{kl}^{(t)}}{\pi_{k+}^{(t)}} n_{10k+} + \frac{\pi_{kl}^{(t)}}{\pi_{+l}^{(t)}} n_{01+l} + \pi_{kl}^{(t)} n_{00++} \right),$$

where

$$n_{11kl} = \sum_i W_{ikl} I(r_i = 11), \quad n_{10k+} = \sum_i W_{ik+} I(r_i = 10),$$
$$n_{01+l} = \sum_i W_{i+l} I(r_i = 01), \quad n_{00++} = \sum_i I(r_i = 00)$$

This combines E and M into a single step!

R Time!

Open file `Lecture07code.R`

Summary

Main take-aways from today's lecture:

- Example of EM algorithm for categorical variables
- Bootstrap confidence intervals

Next lecture:

- Introduction to Bayesian inference! (why??)