Statistical Methods for Analysis with Missing Data

Lecture 2: general setup, notation, missing-data mechanisms

Mauricio Sadinle

Department of Biostatistics

W UNIVERSITY of WASHINGTON
Previous Lecture

\[ p(y) = p(y \mid R = 0)p(R = 0) + p(y \mid R = 1)p(R = 1) \]

what we want

what we can get

We cannot recover \( p(y \mid R = 0) \) nor \( p(y) \) from observed data alone

The fundamental problem of inference with missing data: it is impossible without extra, usually untestable, assumptions on how missingness arises
Today's Lecture

- General setup, notation
- Missing-data mechanisms

Reading: pages 14 – 22, Ch. 1, of Davidian and Tsiatis
Outline

Notation

Missing-Data Mechanisms
### Study Variables and Response Indicators

<table>
<thead>
<tr>
<th>Gender</th>
<th>Age</th>
<th>Income</th>
<th>$R_{Gender}$</th>
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Study Variables and Response Indicators

- $Z = (Z_1, \ldots, Z_K)$: the study variables, or the variables that we intend to measure on each individual
  - Each $Z_k$, $k = 1, \ldots, K$, is a block of variables that are jointly missing/observed

- $R = (R_1, \ldots, R_K)$: the response indicators
  - Each $R_k$, $k = 1, \ldots, K$, is an indicator of whether $Z_k$ is observed
    \[ R_k = \begin{cases} 1 & \text{if } Z_k \text{ is observed,} \\ 0 & \text{if } Z_k \text{ is missing.} \end{cases} \]
Study Variables and Response Indicators

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Sample Data

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For each individual $i = 1, \ldots, n$, we define

- **Study variables:** $Z_i = (Z_{i1}, \ldots, Z_{iK})$
- **Response indicators:** $R_i = (R_{i1}, \ldots, R_{iK})$
Sample Data

- We assume the full sample are independent and identically distributed (i.i.d.) draws

\[ \{(Z_i, R_i)\}_{i=1}^n \sim F \]

from some distribution \( F \)

- Of course, this an idealized scenario: we typically cannot fully observe \( Z_i \)

- In this lecture, we delete the subindex \( i \) to talk about a generic draw from \( F \)
Response and Missingness Patterns

- Each of the components of $Z$ can either be missing or observed, so in general

$$R = (R_1, \ldots, R_K) \in \{0, 1\}^K$$

*Example:* if $K = 2$, $\{0, 1\}^2 = \{(0,0), (1,0), (0,1), (1,1)\}$

- $r = (r_1, \ldots, r_K)$: generic element of $\{0, 1\}^K$, a *response pattern*
  - Sometimes we write $r$ as a string $r = r_1 \ldots r_K$
  - e.g., $r = (0, 1, 0) \equiv 010$

- $\bar{R} = (1 - R_1, \ldots, 1 - R_K)$: the *missingness indicators*

- $\bar{r}$: generic value of $\bar{R}$, a *missingness pattern*
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- \(\bar{r}\): generic value of \(\bar{R}\), a missingness pattern
Notation Example: Regression

Say

\[ Z = (Y, X) = (Y, X_1, \ldots, X_p) \]

where \( Y \) is a response, and \( X \) are covariates

- Say only the outcome \( Y \) can be missing, then
  - \( Z = (Z_1, Z_2), \quad Z_1 = Y, \quad Z_2 = X \)
  - \( R = (R_1, R_2) \in \{(0, 1), (1, 1)\} \)
  - Alternatively, we could define \( R \in \{0, 1\}, \ R = 1 \) if \( Y \) is observed

- Say outcome \( Y \) and covariates \( X \) can be missing (all covariates at the same time), then
  - \( Z = (Z_1, Z_2), \quad Z_1 = Y, \quad Z_2 = X \)
  - \( R = (R_1, R_2) \in \{0, 1\}^2 \)

- Say outcome \( Y \) and individual covariates \( X_1, \ldots, X_p \) can be missing (regardless of the missing status of others), then
  - \( Z = (Z_1, Z_2, \ldots, Z_{p+1}), \quad Z_1 = Y, \quad Z_2 = X_1, \quad \ldots, \quad Z_{p+1} = X_p \)
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  - \( R = (R_1, \ldots, R_{p+1}) \in \{0, 1\}^{p+1} \)
Study participants’ characteristics are to be measured at $T$ times

- $Z_j$: measurements taken at time $t_j$
- $R_j$: indicator of whether participant shows up at time $t_j$

If missingness only comes from subjects dropping out

- Drop out at time $t_j$: $Z_1, \ldots, Z_{j-1}$ observed; $Z_j, \ldots, Z_T$ not observed
- $R = (R_1, \ldots, R_T) \in \{(1, 0, \ldots, 0), (1, 1, 0, \ldots, 0), \ldots, (1, 1, \ldots, 1)\}$
- Can be uniquely summarized by the drop out time $D = 1 + \sum_{j=1}^{T} R_j$

If participants sporadically show up

- $R = (R_1, \ldots, R_T) \in \{0, 1\}^T$
Study participants’ characteristics are to be measured at $T$ times

- $Z_j$: measurements taken at time $t_j$

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- If participants sporadically show up
  - $R = (R_1, \ldots, R_T) \in \{0, 1\}^T$
Notation Example: Longitudinal Study

Study participants’ characteristics are to be measured at $T$ times

▶ $Z_j$: measurements taken at time $t_j$

▶ $R_j$: indicator of whether participant shows up at time $t_j$

▶ If missingness only comes from subjects dropping out
  ▶ Drop out at time $t_j$: $Z_1, \ldots, Z_{j-1}$ observed; $Z_j, \ldots, Z_T$ not observed
  ▶ $R = (R_1, \ldots, R_T) \in \{(1, 0, \ldots, 0), (1, 1, 0, \ldots, 0), \ldots, (1, 1, \ldots, 1)\}$
  ▶ Can be uniquely summarized by the drop out time $D = 1 + \sum_{j=1}^{T} R_j$

▶ If participants sporadically show up
  ▶ $R = (R_1, \ldots, R_T) \in \{0, 1\}^T$
Missing and Observed Data

Given $R = r$

- $Z_r$: observed values
- $Z_{\overline{r}}$: missing values

Example:

- $Z = (Z_1, Z_2, Z_3)$
- If $r = 010$, $Z_r = Z_{010} = Z_2$, and $Z_{\overline{r}} = Z_{101} = (Z_1, Z_3)$

HW1: write down $Z_r$ and $Z_{\overline{r}}$ for all possible values of $r \in \{0, 1\}^3$
Missing and Observed Data

Given \( R = r \)

- \( Z(r) \): observed values
- \( Z(\bar{r}) \): missing values

**Example:**

- \( Z = (Z_1, Z_2, Z_3) \)

- If \( r = 010 \), \( Z(r) = Z(010) = Z_2 \), and \( Z(\bar{r}) = Z(101) = (Z_1, Z_3) \)

**HW1:** write down \( Z(r) \) and \( Z(\bar{r}) \) for all possible values of \( r \in \{0, 1\}^3 \)
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- \( Z(r) \): observed values
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HW1: write down \( Z(r) \) and \( Z(\bar{r}) \) for all possible values of \( r \in \{0, 1\}^3 \)
Observe Data

Given that $R$ is random, the observed data are obtained as realizations of

$$(Z(R), R)$$

We can think of the generative process

$$Z \longrightarrow R \longrightarrow (Z(R), R)$$

 HW1: explain what is the difference between $(Z(R), R)$ and $(Z(r), R = r)$ for a fixed value $r$

 HW1:

a) say $Z = (Z_1, Z_2)$, $Z_1 \in \{1, 2\}$, $Z_2 \in \{A, B\}$, $R \in \{0, 1\}^2$. Write down all the elements of the sample space of $(Z(R), R)$.

b) Describe the sample space of $(Z(R), R)$ if instead $Z \in \mathbb{R}^2$. 
Observed Data

Given that $R$ is random, the observed data are obtained as realizations of

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We can think of the generative process

$$Z \implies R \implies (Z(R), R)$$

- **HW1**: explain what is the difference between $(Z(R), R)$ and $(Z_r, R = r)$ for a fixed value $r$

- **HW1**:
  a) say $Z = (Z_1, Z_2)$, $Z_1 \in \{1, 2\}$, $Z_2 \in \{A, B\}$, $R \in \{0, 1\}^2$. Write down all the elements of the sample space of $(Z(R), R)$.

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▶ **HW1:**

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b) Describe the sample space of $(Z(R), R)$ if instead $Z \in \mathbb{R}^2$. 
Given that $R$ is random, the observed data are obtained as realizations of

$$(Z_{(R)}, R)$$

We can think of the generative process

$$Z \implies R \implies (Z_{(R)}, R)$$

**HW1:** explain what is the difference between $(Z_{(R)}, R)$ and $(Z_{(r)}, R = r)$ for a fixed value $r$

**HW1:**

a) say $Z = (Z_1, Z_2)$, $Z_1 \in \{1, 2\}$, $Z_2 \in \{A, B\}$, $R \in \{0, 1\}^2$. Write down all the elements of the sample space of $(Z_{(R)}, R)$.

b) Describe the sample space of $(Z_{(R)}, R)$ if instead $Z \in \mathbb{R}^2$. 
The \((Z_{obs}, Z_{mis})\) Notation

To formally characterize the observed data we need to use the response vector \(R\).

Yet, a large portion of the literature on missing data define the observed and missing data as

\[
Z = (Z_{obs}, Z_{mis})
\]

- \(Z_{obs}\): observed values, so \(Z_{obs} = Z(R)\)
- \(Z_{mis}\): missing values, so \(Z_{mis} = Z(\bar{R})\)

This notation is convenient for its simplicity, but in this course we avoid it, as \(Z_{obs}\) and \(Z_{mis}\) do not explicitly indicate how they relate to \(R\).
If missingness comes only from subjects dropping out

- Missingness patterns are uniquely summarized by the drop out time
  \[ D = 1 + \sum_{j=1}^{T} R_j \]

- The observed data are obtained as realizations of
  \[ (Z_{(D)}, D) \]
  where, if \( D = d \), \( Z_{(d)} = (Z_1, \ldots, Z_{d-1}) \)
Distributions of Interest

- Full-data distribution: joint distribution of \((Z, R)\)
  - Density: \(p(z, r) = p(z \mid r)p(r) = p(r \mid z)p(z)\)

- Davidian and Tsiatis refer to the distribution of \(Z\) as the full-data distribution, but \(R\) is also data!

- Missing-data mechanism or missingness mechanism: conditional distribution of \(R \mid Z\)
  - Density: \(p(r \mid z)\)
Distributions of Interest

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- Missing-data mechanism or missingness mechanism: conditional distribution of \(R \mid Z\)
  - Density: \(p(r \mid z)\)
Notation for Density Functions

For simplicity we use $p(\cdot)$ for technically different functions

- $p(z) \equiv p_Z(z)$
- $p(z, r) \equiv p_{Z,R}(z, r)$
- $p(r \mid z) \equiv p_{R \mid Z}(r \mid z)$

Interpretations should be clear from the arguments passed to them
Outline

Notation

Missing-Data Mechanisms
Missing-Data Mechanisms: A Bit of History

- Missing data was largely seen as a computational issue: “these holes in the data don’t let me run my analysis”

- The inferential complications induced by missing data were first addressed in a seminal paper by Rubin (1976, Biometrika)

- Prior to this, some authors had ways of “ignoring” the missing data, but no formal treatment of the missingness mechanism existed

- The definitions that Rubin introduced have evolved: see lectures on likelihood-based inference
We’ll introduce the classification of missing-data mechanisms as they are commonly interpreted, and as presented by Davidian and Tsiatis.

However, as we’ll see in the lectures on likelihood-based inference, this is not exactly the interpretation that Rubin intended.
Missing-Data Mechanisms: Missing Completely at Random

Data are said to be *missing completely at random* (MCAR) if

\[ p(R = r \mid z) = p(R = r) \]

Interpreted as

- \( R \perp \perp Z \) \hspace{0.5cm} (\( R \) and \( Z \) are independent)
- Missingness has nothing to do with the study variables
Missing-Data Mechanisms: Missing Completely at Random

MCAR: \( p(R = r \mid z) = p(R = r) \)

Example:

let’s say \( Z = (\text{Sex}, \text{Age}, \text{Income}) \)

▶ Say \( r = 110 \),

\[
p(R = 110 \mid M, 25, 10K) = p(R = 110 \mid F, 70, 60K) = p(R = 110)
\]

▶ Same for all other response patterns \( r \)

▶ We conclude

\[ R \perp \perp (\text{Sex}, \text{Age}, \text{Income}) \]
Missing-Data Mechanisms: Missing Completely at Random

MCAR: \( p(R = r \mid z) = p(R = r) \)

*Example:*

let's say \( Z = (\text{Sex, Age, Income}) \)

- Say \( r = 110 \),
  \[
  p(R = 110 \mid M, 25, 10K) = p(R = 110 \mid F, 70, 60K) = p(R = 110)
  \]

- Same for all other response patterns \( r \)

- We conclude
  \[ R \perp \perp (\text{Sex, Age, Income}) \]
Missing-Data Mechanisms: Missing Completely at Random

MCAR:  \[ p(R = r | z) = p(R = r) \]

Example:

let's say \( Z = (Sex, Age, Income) \)

- Say \( r = 110 \),
  \[ p(R = 110 | M, 25, 10K) = p(R = 110 | F, 70, 60K) = p(R = 110) \]

- Same for all other response patterns \( r \)

- We conclude
  \( R \perp \perp (Sex, Age, Income) \)
Missing-Data Mechanisms: Missing Completely at Random

MCAR: \[ p(R = r \mid z) = p(R = r) \]

*Example:*

let's say \( Z = (\text{Sex}, \text{Age}, \text{Income}) \)

- Say \( r = 110 \),
  \[ p(R = 110 \mid M, 25, 10K) = p(R = 110 \mid F, 70, 60K) = p(R = 110) \]

- Same for all other response patterns \( r \)

- We conclude
  \[ R \independent (\text{Sex}, \text{Age}, \text{Income}) \]
Data are said to be *missing at random* (MAR) if

\[ p(R = r \mid z) = p(R = r \mid z(r)) \]

Interpreted as

- The probability of a response pattern does not depend on the missing data
- The probability of response pattern \( r \) as a function of \( z \) is constant on \( z(\bar{r}) \)
Missing-Data Mechanisms: Missing at Random

MAR: \( p(R = r \mid z) = p(R = r \mid z(r)) \)

Example:

let’s say \( Z = (Sex, Age, Income) \), and only income can be missing

- If \( r = 110 \),
  \[
p(R = 110 \mid z) = p(R = 110 \mid z_{(110)}) = p(R = 110 \mid Sex, Age)
  \]

- If \( r = 111 \),
  \[
p(R = 111 \mid z) = p(R = 111 \mid z_{(111)}) = p(R = 111 \mid Sex, Age, Income)
  \]

- However, since only income can be missing,
  \[
p(R = 111 \mid z) = 1 - p(R = 110 \mid z)
  \]

- Therefore \( p(R = 111 \mid z) = p(R = 111 \mid Sex, Age) \) and we conclude

\[
R \perp \perp Income \mid Sex, Age
\]

- (Here we could simply define \( R \) as the indicator of missingness for \( Income \)
Missing-Data Mechanisms: Missing at Random

**MAR:** \( p(R = r \mid z) = p(R = r \mid z(r)) \)

**Example:**

Let’s say \( Z = (\text{Sex}, \text{Age}, \text{Income}) \), and only income can be missing

- If \( r = 110 \),
  \[
  p(R = 110 \mid z) = p(R = 110 \mid z(110)) = p(R = 110 \mid \text{Sex}, \text{Age})
  \]

- If \( r = 111 \),
  \[
  p(R = 111 \mid z) = p(R = 111 \mid z(111)) = p(R = 111 \mid \text{Sex}, \text{Age}, \text{Income})
  \]

- However, since only income can be missing,
  \[
  p(R = 111 \mid z) = 1 - p(R = 110 \mid z)
  \]

- Therefore \( p(R = 111 \mid z) = p(R = 111 \mid \text{Sex}, \text{Age}) \) and we conclude
  \[
  R \perp \perp \text{Income} \mid \text{Sex, Age}
  \]

- (Here we could simply define \( R \) as the indicator of missingness for \( \text{Income} \))
Missing-Data Mechanisms: Missing at Random

**MAR:** \[ p(R = r \mid z) = p(R = r \mid z(r)) \]

**Example:**

let's say \( Z = (Sex, Age, Income) \), and only income can be missing

- If \( r = 110 \),
  \[ p(R = 110 \mid z) = p(R = 110 \mid z_{(110)}) = p(R = 110 \mid Sex, Age) \]

- If \( r = 111 \),
  \[ p(R = 111 \mid z) = p(R = 111 \mid z_{(111)}) = p(R = 111 \mid Sex, Age, Income) \]

- However, since only income can be missing,
  \[ p(R = 111 \mid z) = 1 - p(R = 110 \mid z) \]

- Therefore \( p(R = 111 \mid z) = p(R = 111 \mid Sex, Age) \) and we conclude
  \[ R \perp\perp Income \mid Sex, Age \]

- (Here we could simply define \( R \) as the indicator of missingness for \( Income \))
Missing-Data Mechanisms: Missing at Random

\[
\text{MAR: } p(R = r \mid z) = p(R = r \mid z_{(r)})
\]

Example:

let’s say \( Z = (\text{Sex, Age, Income}) \), and only income can be missing

- If \( r = 110 \),

\[
p(R = 110 \mid z) = p(R = 110 \mid z_{(110)}) = p(R = 110 \mid \text{Sex, Age})
\]

- If \( r = 111 \),

\[
p(R = 111 \mid z) = p(R = 111 \mid z_{(111)}) = p(R = 111 \mid \text{Sex, Age, Income})
\]

- However, since only income can be missing,

\[
p(R = 111 \mid z) = 1 - p(R = 110 \mid z)
\]

- Therefore \( p(R = 111 \mid z) = p(R = 111 \mid \text{Sex, Age}) \) and we conclude

\[
R \perp \perp \text{Income} \mid \text{Sex, Age}
\]

- (Here we could simply define \( R \) as the indicator of missingness for \( \text{Income} \)).
Missing-Data Mechanisms: Missing at Random

**MAR:** \( p(R = r \mid z) = p(R = r \mid z_{(r)}) \)

**Example:**

let's say \( Z = (\text{Sex}, \text{Age}, \text{Income}) \), and only income can be missing

- If \( r = 110 \),
  \[
  p(R = 110 \mid z) = p(R = 110 \mid z_{(110)}) = p(R = 110 \mid \text{Sex}, \text{Age})
  \]

- If \( r = 111 \),
  \[
  p(R = 111 \mid z) = p(R = 111 \mid z_{(111)}) = p(R = 111 \mid \text{Sex}, \text{Age}, \text{Income})
  \]

- However, since only income can be missing,
  \[
  p(R = 111 \mid z) = 1 - p(R = 110 \mid z)
  \]

- Therefore \( p(R = 111 \mid z) = p(R = 111 \mid \text{Sex}, \text{Age}) \) and we conclude
  \[
  R \perp \perp \text{Income} \mid \text{Sex}, \text{Age}
  \]

  (Here we could simply define \( R \) as the indicator of missingness for \( \text{Income} \))
Missing-Data Mechanisms: Missing at Random

MAR: \[ p(R = r \mid z) = p(R = r \mid z_{(r)}) \]

Example:

let’s say \( Z = (Sex, Age, Income) \), and only income can be missing

- If \( r = 110 \),
  \[ p(R = 110 \mid z) = p(R = 110 \mid z_{(110)}) = p(R = 110 \mid Sex, Age) \]

- If \( r = 111 \),
  \[ p(R = 111 \mid z) = p(R = 111 \mid z_{(111)}) = p(R = 111 \mid Sex, Age, Income) \]

- However, since only income can be missing,
  \[ p(R = 111 \mid z) = 1 - p(R = 110 \mid z) \]

- Therefore \( p(R = 111 \mid z) = p(R = 111 \mid Sex, Age) \) and we conclude
  \[ R \perp \perp Income \mid Sex, Age \]

- (Here we could simply define \( R \) as the indicator of missingness for \( Income \))
Missing-Data Mechanisms: Missing at Random

**MAR:** \[ p(R = r \mid z) = p(R = r \mid z(r)) \]

*Example:*

let’s say \( Z = (\text{Sex}, \text{Age}, \text{Income}) \), and any missingness pattern is possible

- If \( r = 110 \),
  \[ p(R = 110 \mid z) = p(R = 110 \mid z(110)) = p(R = 110 \mid \text{Sex}, \text{Age}) \]

- If \( r = 111 \),
  \[ p(R = 111 \mid z) = p(R = 111 \mid z(111)) = p(R = 111 \mid \text{Sex}, \text{Age}, \text{Income}) \]

- If \( r = 001 \),
  \[ p(R = 001 \mid z) = p(R = 001 \mid z(001)) = p(R = 001 \mid \text{Income}) \]

- If \( r = 000 \),
  \[ p(R = 000 \mid z) = p(R = 000 \mid z(000)) = p(R = 000) \]

- How do you like this as an assumption?
Missing-Data Mechanisms: Missing at Random

MAR: \[ p(R = r \mid z) = p(R = r \mid z_r) \]

**Example:**

let’s say \( Z = (Sex, Age, Income) \), and any missingness pattern is possible

- If \( r = 110 \),
  \[ p(R = 110 \mid z) = p(R = 110 \mid z_{110}) = p(R = 110 \mid Sex, Age) \]

- If \( r = 111 \),
  \[ p(R = 111 \mid z) = p(R = 111 \mid z_{111}) = p(R = 111 \mid Sex, Age, Income) \]

- If \( r = 001 \),
  \[ p(R = 001 \mid z) = p(R = 001 \mid z_{001}) = p(R = 001 \mid Income) \]

- If \( r = 000 \),
  \[ p(R = 000 \mid z) = p(R = 000 \mid z_{000}) = p(R = 000) \]

- How do you like this as an assumption?
Missing-Data Mechanisms: Missing at Random

MAR: \( p(R = r \mid z) = p(R = r \mid z_{(r)}) \)

Example:

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- If \( r = 110 \),
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  \[
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- How do you like this as an assumption?
Missing-Data Mechanisms: Missing at Random

**MAR:** \( p(R = r \mid z) = p(R = r \mid z_{(r)}) \)

*Example:*

let’s say \( Z = (\text{Sex}, \text{Age}, \text{Income}) \), and any missingness pattern is possible

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- If \( r = 111 \),
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p(R = 111 \mid z) = p(R = 111 \mid z_{(111)}) = p(R = 111 \mid \text{Sex, Age, Income})
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  \]

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  \[
p(R = 000 \mid z) = p(R = 000 \mid z_{(000)}) = p(R = 000)
  \]

- How do you like this as an assumption?
Missing-Data Mechanisms: Missing at Random

\[ \text{MAR: } p(R = r \mid z) = p(R = r \mid z(r)) \]

**Example:**

let’s say \( Z = (\text{Sex}, \text{Age}, \text{Income}) \), and any missingness pattern is possible

- If \( r = 110 \),
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- How do you like this as an assumption?
Missing-Data Mechanisms: Missing at Random

**MAR:** \[ p(R = r \mid z) = p(R = r \mid z(r)) \]

**Example:** say \( Z = (Z_1, Z_2), \quad (R_1, R_2) \in \{0, 1\}^2 \)

- \[ p(R_1 = 0, R_2 = 0 \mid Z_1 = z_1, Z_2 = z_2) = f_{00} \]
- \[ p(R_1 = 1, R_2 = 0 \mid Z_1 = z_1, Z_2 = z_2) = f_{10}(z_1) \]
- \[ p(R_1 = 0, R_2 = 1 \mid Z_1 = z_1, Z_2 = z_2) = f_{01}(z_2) \]
- \[ p(R_1 = 1, R_2 = 1 \mid Z_1 = z_1, Z_2 = z_2) = 1 - f_{00} - f_{10}(z_1) - f_{01}(z_2) \]

So MAR in general is NOT a conditional independence statement!
Missing-Data Mechanisms: Missing at Random

MAR: \[ p(R = r \mid z) = p(R = r \mid z(r)) \]

Example: say \( Z = (Z_1, Z_2) \), \( (R_1, R_2) \in \{0, 1\}^2 \)

- \( p(R_1 = 0, R_2 = 0 \mid Z_1 = z_1, Z_2 = z_2) = f_{00} \)
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- \[ p(R_1 = 0, R_2 = 1 \mid Z_1 = z_1, Z_2 = z_2) = f_{01}(z_2) \]
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So MAR in general is NOT a conditional independence statement!
Missing-Data Mechanisms: Missing Not at Random

Data are said to be *missing not at random* (MNAR) if

\[ p(R = r \mid z) \neq p(R = r \mid z(r)) \]

- Quite literally, anything that cannot be written as MAR
- The probability of observing \( r \) depends on the components of \( Z \) not observed when \( R = r \)
- Probably the most realistic scenario, and the most difficult to handle
A Toy Example

- $Y \in \{0, 1\}$: indicates presence of a feature, sometimes missing
- $X \in \{A, B\}$: population groups, always observed
- $R \in \{0, 1\}$: response indicator for $Y$
A Toy Example: Full Data

\[ p(R = 1 \mid x, y) = 1 \]
A Toy Example: Missing Completely at Random

\[ p(R = 1 \mid x, y) = p(R = 1) = 0.8 \]
A Toy Example: Missing at Random

\[ p(R = 1 \mid x, y) = p(R = 1 \mid x) = 0.8I(x = A) + 0.4I(x = B) \]
A Toy Example: Missing Not at Random

\[ p(R = 1 \mid x, y) = p(R = 1 \mid y) = 0.8I(y = 0) + 0.2I(y = 1) \]
What Can We Conclude So Far?

- In general, missing data complicates inference
- In the scale of complication

\[ MCAR \ll MAR \ll MNAR \]

- But how can we know?
  - MCAR vs MAR?: doable, but relies on assumption that MAR holds
  - MAR vs MNAR?: not possible based on your observed data – MNAR mechanisms depend on data that are not observed
  - The data analyst must adopt an assumption about the mechanism without being able to verify it

- “if one adopts an assumption of MAR, it must be defensible on scientific, subject matter, and/or practical grounds, because it cannot be validated from the data” Davidian and Tsiatis

- Inference under MNAR is more realistic but more complicated – we’ll look into this towards the end of the course
What Can We Conclude So Far?

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- Inference under MNAR is more realistic but more complicated – we’ll look into this towards the end of the course
Never Work Under MAR?

Most approaches for inference with missing data assume MAR

- Option 1: “don’t worry about how the sausage gets made, just eat the sausage!,” or the approach of the horse with blinders:
Never Work Under MAR?

Most approaches for inference with missing data assume MAR

- Option 2: you can argue that MAR is not unreasonable. For example, do you have sufficiently rich information that is always observed?
  - Say \( Z = (Z_1, Z_2) \)
  - \( Z_1 \): a vector subject to missingness
  - \( Z_2 \): fully observed
  - \( R \): response indicator for \( Z_1 \)
  - MAR: \( p(R = r \mid z_1, z_2) = p(R = r \mid z_{1(r)}, z_2) \)
  - If assuming \( p(R = r \mid z_1, z_2) = p(R = r \mid z_2) \) is reasonable, then MAR is reasonable because MAR is more general
Never Work Under MAR?

Most approaches for inference with missing data assume MAR

- Option 3: take this class, think about these issues, contribute to creating better solutions!
Summary

Main take-aways from today’s lecture:

▶ Proper handling of missing data requires proper notation
▶ Universe of missing-data assumptions:

Next lecture:

▶ Naïve methods for handling missing data: imputation and complete cases
▶ Reading: Chapter 2 in Davidian and Tsiatis
Summary

Main take-aways from today’s lecture:

- Proper handling of missing data requires proper notation
- Universe of missing-data assumptions:
  - MAR
  - MCAR
  - MNAR

Next lecture:

- Naïve methods for handling missing data: imputation and complete cases
- Reading: Chapter 2 in Davidian and Tsiatis